Exponentials and Logarithms: Lecture 2

Recall: Definition of e, an alternative approach.

Let $f(x) = \ln x$

$$\frac{d}{dx} \ln x \Big|_{x=1} = \frac{1}{x}\Big|_{x=1} = 1$$

$$\therefore f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$1 = \lim_{h \to 0} \frac{\ln(1+h) - \ln 1}{h}$$

Recall: $\ln 1 = 0$

$$1 = \lim_{h \to 0} \frac{\ln(1+h) - 0}{h}$$

$$1 = \lim_{h \to 0} \frac{1}{h} \ln(1+h)$$

$$1 = \lim_{h \to 0} \ln(1+h)^{1/h}$$

(By the Rules of Logarithms, $\ln x^n = n \ln x$)

$$1 = \ln\left[\lim_{h \to 0} (1+h)^{1/h}\right]$$

(The log of the limit = limit of the log)

$$\therefore \boxed{e = e^{\ln\left[\lim_{h \to 0}(1+h)^{1/h}\right]} = \lim_{h \to 0} (1+h)^{1/h}}$$

$d/dx \ln x = 1/x$

$d/dx\, e^{kx} = ke^{kx}$

$d/dx\, e^x = e^x$

$d/dx\, e^{f(x)} = e^{f(x)} \cdot f'(x)$ (By Chain Rule)

Ex 1:

$y = e^{-5x}$, $dy/dx = ?$

Let $u = -5x$, $du/dx = -5$

$\therefore y = e^{u}$, $dy/du = e^{u}$

Chain Rule

$\therefore dy/dx = dy/du \cdot du/dx$

$= e^{u} \cdot (-5)$

$= \boxed{-5e^{-5x}}$

STYLE: How you write and show your work through a problem reflects your thinking and writing style. This is very closely connected to the clarity of your strategy and solution. Writing out dy/dx, dy/du and du/dx as shown immediately shows us the relevance of the chain rule in the derivative.

Ex 2:

$y = e^{x^2}$, $dy/dx = ?$

Let $u = x^2$, $du/dx = 2x$

$\therefore y = e^{u}$, $dy/du = e^{u}$

$\therefore dy/dx = dy/du \cdot du/dx$

$= e^{u} \cdot 2x$

$= \boxed{2xe^{x^2}}$

Ex 3:

$y = e^{2x/3}$, $dy/dx = ?$

Let $u = \frac{2x}{3}$, $du/dx = \frac{2}{3}$

$\therefore y = e^{u}$, $dy/du = e^{u}$

$\therefore dy/dx = dy/du \cdot du/dx$

$= e^{u} \cdot \frac{2}{3}$

$= \boxed{\frac{2}{3}e^{2x/3}}$

Ex4 Using the Product Rule:

PRODUCT RULE

$$f'(u \cdot v) = v\,du/dx + u\,dv/dx$$

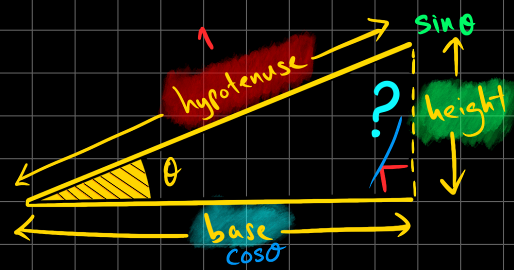
$$y = \underset{u\ v}{xe^x} - e^x$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx} - \frac{d}{dx}e^x$$

$$= \cancel{e^x}(1) + x \cdot e^x - \cancel{e^x}$$

$$= \boxed{xe^x}$$

Ex5: With Trigonometric Ratios



$$y = \underset{u}{e^\theta}\underset{v}{(\sin\theta + \cos\theta)}$$

STRATEGY NOTE!

Multiplying through will create two terms, each requiring use of the Product Rule. But, since e^θ has a simple derivative expression, it is best to use it as an easy, strategic substitution for 'u'. Also, we have an easy 'v'.

$$\frac{dy}{d\theta} = v\frac{du}{d\theta} + u\frac{dv}{d\theta}$$

$$= (\sin\theta + \cos\theta) \cdot e^\theta + e^\theta \cdot (\cos\theta - \sin\theta)$$

$$= e^\theta(\cancel{\sin\theta} + \cos\theta - \cancel{\sin\theta} + \cos\theta) = \boxed{e^\theta\, 2\cos\theta}$$

Ex 6: With Natural Logarithm

ln x

$$y = \ln(3\theta e^{-\theta}) \quad , \quad dy/d\theta = ?$$

Let $u = 3\theta e^{-\theta}$, $du/d\theta = v\dfrac{du}{d\theta} + u\dfrac{dv}{d\theta}$

$u \quad v$

$$= e^{-\theta}(3) + 3\theta(-e^{-\theta})$$

$$= 3e^{-\theta} - 3\theta e^{-\theta}$$

$$= 3e^{-\theta}(1-\theta)$$

$\therefore \quad y = \ln u \quad , \quad dy/du = \dfrac{1}{u}$

$\therefore \quad dy/d\theta = dy/du \cdot du/d\theta$

$$= \frac{1}{u} \cdot 3e^{-\theta}(1-\theta)$$

$$= \frac{\cancel{3}\cancel{e^{-\theta}}(1-\theta)}{\cancel{3}\theta \cancel{e^{-\theta}}}$$

$$= \frac{1-\theta}{\theta}$$

$$= \boxed{\frac{1}{\theta} - 1}$$

Ex 7:

$y = \cos\left(e^{-\theta^2}\right)$, $dy/d\theta = ?$

Let $u = -\theta^2$, $du/d\theta = -2\theta$

$\therefore y = \cos(e^u)$, $dy/du = -\sin e^u \cdot d/du\, e^u$

$= -e^u \sin e^u$

$\therefore dy/d\theta = dy/du \cdot du/d\theta$

$= -e^u \sin e^u \cdot (-2\theta)$

$= 2\theta e^u \sin e^u$

$= \boxed{2\theta e^{-\theta^2} \sin e^{-\theta^2}}$

Ex 8: $y = \pi^x$

Since e and $\ln$ are inverse functions

by the laws of logarithmic operations.

$\frac{d}{dx}\pi^x = \frac{d}{dx}e^{\ln \pi^x} = \frac{d}{dx}e^{x\ln\pi}$

$\therefore \frac{d}{dx}e^{x\ln\pi} = e^{x\ln\pi} \cdot \frac{d}{dx}x\ln\pi$

$= e^{x\ln\pi} \cdot \ln\pi = e^{\ln\pi^x} \cdot \ln\pi = \boxed{\pi^x \ln\pi}$

The "Weird" Ones:

Ex 9:

$$y = x^{\sin x}$$

$$\ln y = \ln x^{\sin x} = \sin x \ln x$$

Taking the logarithm on both sides.

$$\frac{d}{dx} \ln y = \frac{d}{dx}(\underset{u}{\sin x}\,\underset{v}{\ln x})$$

Chain Rule

$$\frac{d}{dy} \ln y \cdot \frac{dy}{dx} = v\, du/dx + u\, dv/dx$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \ln x \cdot \cos x + \sin x \cdot \frac{1}{x}$$

$$= \cos x \cdot \ln x + \frac{\sin x}{x}$$

$$\therefore \frac{dy}{dx} = y\left(\cos x \ln x + \frac{\sin x}{x}\right)$$

$$= \boxed{x^{\sin x}\left[\cos x \cdot \ln x + \frac{\sin x}{x}\right]}$$

Ex 10

$y = \ln x^{\ln x}$

$\therefore \ln y = \ln\left(\ln x^{\ln x}\right)$ Taking the logarithm on both sides

$\ln y = \ln x \cdot \ln\ln x$ By the Laws of Logarithms

$\therefore \frac{d}{dx}\ln y = \frac{d}{dx}[\underset{u}{\ln x} \cdot \underset{v}{\ln\ln x}]$

$u = \ln x \quad , \; du/dx = \frac{1}{x}$

Chain Rule

$v = \ln(\ln x) \; , \; dv/dx = \frac{1}{\ln x} \cdot \frac{d}{dx}\ln x = \frac{1}{\ln x} \cdot \frac{1}{x} = \frac{1}{x\ln x}$

$[f(g(x))]' = f'(g(x))\, g'(x)$

$\frac{d}{dy}\ln y \cdot \frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$

$\frac{1}{y} \cdot \frac{dy}{dx} = \ln\ln x \cdot \left(\frac{1}{x}\right) + \cancel{\ln x} \cdot \frac{1}{x\cancel{\ln x}}$

$= \frac{\ln\ln x}{x} + \frac{1}{x}$

$= \frac{1}{x}(\ln\ln x + 1)$

$\therefore \frac{dy}{dx} = y \cdot \frac{1}{x}(\ln\ln x + 1)$

$= \ln x^{\ln x} \cdot \frac{1}{x}(\ln\ln x + 1) = \frac{1}{x}\ln x^{\ln x} \cdot (\ln\ln x + 1)$

Ex11: The Triple Product ×3

$$y = \underbrace{\theta^3 e^{-2\theta}}_{u} \cos 5\theta \quad \checkmark$$

STRATEGY NOTE!

Here, we revisit the triple product problem. A good strategic choice can make all the difference on your style, organization, and chances of successfully solving the problem. We should be careful. The obvious choice is to isolate the easy derivative, the θ, leaving behind a tricky product.

One way or another, we are going to need to use the Product Rule at least once later. So we create a split that gives me the easiest execution of the Product Rule. So lets pair the easier ones together.

$$\therefore \ dy/d\theta = d/d\theta \left(\underbrace{\theta^3 e^{-2\theta}}_{u} \cos 5\theta\right) \quad \checkmark$$

$$du/d\theta = d/d\theta \left(\underset{u}{\theta^3} \underset{v}{e^{-2\theta}}\right) = v\,du/dx + u\,dv/dx$$

$$= \underset{v}{e^{-2\theta}} \cdot \underset{du/dx}{3\theta^2} + \underset{u}{\theta^3} \underset{dv/dx}{(-2e^{-2\theta})} = 3\theta^2 e^{-2\theta} - 2\theta^3 e^{-2\theta}$$

$$dv/d\theta = d/d\theta(\cos 5\theta) = -5\sin 5\theta$$

$$\therefore \ dy/d\theta = v\,du/d\theta + u\,dv/d\theta$$

$$= \cos 5\theta \cdot \left(3\theta^2 e^{-2\theta} - 2\theta^3 e^{-2\theta}\right) + \theta^3 e^{-2\theta}(-5\sin 5\theta)$$

$$= \theta^2 e^{-2\theta}\left[\cos 5\theta(3 - 2\theta) - 5\theta^3 e^{-2\theta} \sin 5\theta\right]$$

$$= \boxed{\theta^2 e^{-2\theta}\left(3\cos 5\theta - 2\theta\cos 5\theta - 5\theta \sin 5\theta\right)}$$

Ex 12:

$$y = \frac{(x^2+1)(x+3)^{1/2}}{x-1}$$

$$\ln y = \ln\left[\frac{(x^2+1)(x+3)^{1/2}}{x-1}\right]$$

$$= \ln\left[(x^2+1)(x+3)^{1/2}\right] - \ln(x-1)$$

$$= \ln(x^2+1) + \ln(x+3)^{1/2} - \ln(x-1)$$

$$= \ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left[\ln(x^2+1) + \frac{1}{2}\ln(x+3) - \ln(x-1)\right]$$

$$\frac{d}{dy}\ln y \cdot \frac{dy}{dx} = \frac{1}{x^2+1}\cdot 2x + \frac{1\ (1)}{2(x+3)} + \frac{1\ (-1)}{x-1}$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}$$

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right)$$

$$\frac{dy}{dx} = \boxed{\frac{(x^2+1)(x+3)^{1/2}}{x-1}\left[\frac{2x}{x^2+1} + \frac{1}{2x+6} - \frac{1}{x-1}\right]}$$